Differentiable Stacks III: Stacks

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Motto: Compatible local structures can be glued together.

1 Introduction

Last time, we saw that we could view a category fibred in groupoids as a presheaf with values in groupoids. Therefore, a natural question to ask is when this presheaf is a sheaf. What does this mean? Let $\mathcal{F}\colon \mathcal{C}^{\mathrm{op}} \to \mathsf{Grpd}$ be a presheaf in groupoids. Then we would like the following:[1]

1. Given a covering family $\{f_i\colon C_i \to C\}$ and objects $x_i \in \mathcal{F}(C_i)$ such that $x_i|_{C_{ij}} = x_j|_{C_{ij}}$, then there exists a unique object $x \in \mathcal{F}(C)$ such that $x|_{C_i} = x$.

2. Given a covering family $\{f_i\colon C_i \to C\}$, objects $x, y \in \mathcal{F}(C)$ and 2-morphisms $\phi_i\colon x|_{C_i} \Rightarrow x|_{C_j}$ such that $\phi_i|_{C_{ij}} = \phi_j|_{C_{ij}}$, then there is a unique 2-morphism $\phi\colon x \Rightarrow y$.

Here and in the rest of the document we write, $C_{ij} = C_i \times_C C_j$ and $C_{ijk} = C_i \times_C C_j \times_C C_k$.

[1] The exact gluing properties will be more subtle than this, because $\mathcal{F}$ is a *lax* 2-functor.

2 Descent data

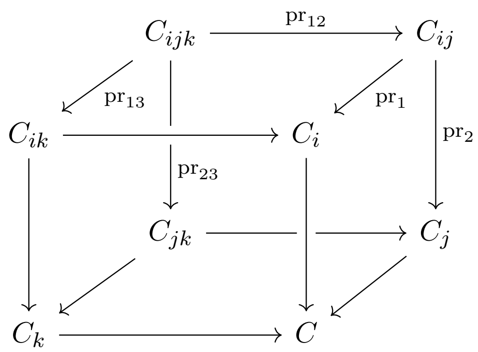
The problem we are interested in is the descent of compatible families of objects in morphisms. Therefore, our first step will be to formalise the input to this descent problem, which is called the descent data.

Definition 2.1 ([6, Def. 4.2])**.** Let $(\mathcal{C}, K)$ be a site and let $\pi\colon \mathcal{D} \to \mathcal{C}$ be a category fibred in groupoids with a chosen cleavage. Let $\mathcal{U} = \{f_i \colon C_i \to C\}$ be a covering family. An *object with descent data* $(\{\xi_i\}, \{\phi_{ij}\})$ on $\mathcal{U}$ is a collection of objects $\xi_i \in \pi_{C_i}$, and a collection of isomorphisms $\phi_{ij}\colon \mathrm{pr}_2^*\xi_j \to \mathrm{pr}_1^*\xi_i$ in $\pi_{C_{ij}}$ such that the cocycle condition

$$\mathrm{pr}_{13}^*\phi_{ik} = \mathrm{pr}_{12}^* \circ \mathrm{pr}_{23}^*\phi_{jk}$$

is satisfied. The descent data is called *normalized* if all $\phi_{ii} = \mathrm{id}$.

The projection maps arise from the pullbacks as indicated here in this diagram.



Since we are working with a category fibred in groupoids, we only ask for compatibility up to an isomorphism, rather than equality on the nose. To do this coherently over a cover, we then require the cocycle condition to be satisfied. One can compare this to defining a vector bundle over a manifold by a local cover and transition functions.

There is also a notion of morphisms of descent data.

Definition 2.2 ([6, Def. 4.2])**.** An *arrow between objects with descent data*

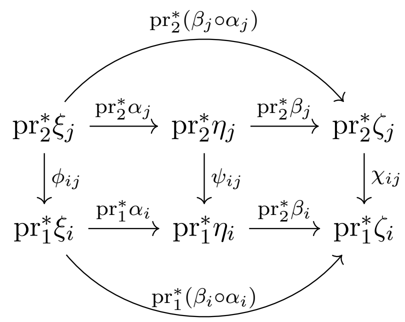
$$\{\alpha_i\}\colon (\{\xi_i\}, \{\phi_{ij}\}) \to (\{\eta_i\}, \{\psi_{ij}\})$$

is a collection of arrows $\alpha_i\colon \xi_i \to \eta_i$ with the property that for each pair of indices i, j the diagram

$$\begin{CD}
\mathrm{pr}_2^*\xi_j @>{\mathrm{pr}_2^*\alpha_j}>> \mathrm{pr}_2^*\eta_j \\
@V{\phi_{ij}}VV @VV{\psi_{ij}}V \\
\mathrm{pr}_1^*\xi_i @>{\mathrm{pr}_1^*\alpha_i}>> \mathrm{pr}_1^*\eta_i
\end{CD}$$

commutes.

The composition of two arrows of objects with descent data is defined as you would expect: Let $\{\alpha_i\}\colon (\{\xi_i\}, \{\phi_{ij}\}) \to (\{\eta_i\}, \{\psi_{ij}\})$ and $\{\beta_i\}\colon (\{\eta_i\}, \{\psi_{ij}\}) \to (\{\zeta_i\}, \{\chi_{ij}\})$ be arrows of objects with descent data. Then we define is composition as $\{\beta_i\} \circ \{\alpha_i\} = \{\beta_i \circ \alpha_i\}$. To see that this is again an arrow of objects with descent data, we need to check that the commutative diagram in the definition commutes. This follows from the fact that pr_i^* are functors:

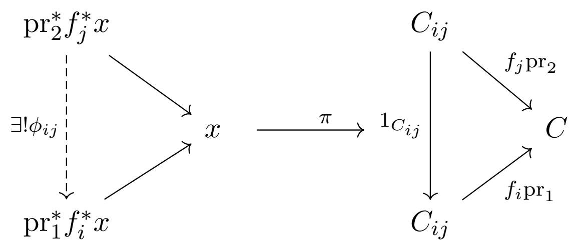


Hence, we see that the descent data corresponding to a covering family $\mathcal{U}$ combines into a category.

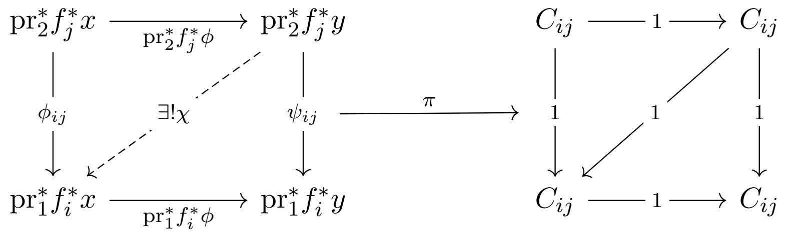
Definition 2.3. The *category of descent data of* $\mathcal{U}$ is the category with objects with descent data as its objects, and arrows of objects with descent data as its morphisms. We denote this category by $\pi_{\mathcal{U}}$.

One can also describe descent data without a choice of cleavage. For details, see [6, pp. 72–75].

Now, we define a functor $D_{\mathcal{U}} \colon \pi_C \to \pi_{\mathcal{U}}$: Let $x \in \pi_C$. Then as our collection of objects, we take the chosen pullbacks $f_i^* x \in \pi_{C_i}$. The second axiom of a category fibred in groupoids gives a unique isomorphism $\phi_{ij} \colon \mathrm{pr}_2^* f_j^* x \to \mathrm{pr}_1^* f_i^* x$, as we see from the following diagram



This yields an object with descent data $(\{f_i^* x\}, \{\phi_{ij}\})$. Given an arrow $\phi \colon x \to y \in \pi_C$, we assign to it the arrow $\{f_i^* \phi\} \colon (\{f_i^* x\}, \{\phi_{ij}\}) \to (\{f_i^* y\}, \{\psi_{ij}\})$. We have to show that this is indeed an arrow of objects with descent data. We deduce this from the following diagram.



By applying the second axiom of a fibred category to the diagonal arrow, we obtain a map χ making the lower right triangle commute. Then applying the second axiom to the upper left triangle, we obtain a unique arrow along the upper horizontal line. But $\mathrm{pr}_2^* f_j^* \phi$ is precisely such an arrow, so we conclude that the upper triangle also commutes, and therefore the outer square commutes.

Lemma 2.4. *The map* $D_{\mathcal{U}} \colon \pi_C \to \pi_{\mathcal{U}}$ *is a functor.*

Proof. This follows from the fact that the composition is defined component wise, and that the pullbacks $f_i^* \colon \pi_C \to \pi_{C_i}$ are functors. $\square$

3 Prestacks and stacks

Definition 3.1 ([6, Def. 4.6])**.** Let $(\mathcal{C}, K)$ be a site and let $\pi\colon \mathcal{D} \to \mathcal{C}$ be a category fibred in groupoids. Then π is a *prestack* if for all objects $C \in \mathcal{C}$ and all covering families $\mathcal{U}$ of C, the functor $\pi_C \to \pi_{\mathcal{U}}$ is fully faithful.

Let us spell out what this means in more concrete terms. Let $C \in \mathcal{C}$, let $\mathcal{U} = \{f_i\colon C_i \to C\}$ be a covering family of C and let $x, y \in \pi_C$. Then the statement that functor is full faithful means that

$$\mathrm{Hom}_{\pi_C}(x, y) \cong \mathrm{Hom}_{\pi_{\mathcal{U}}}((\{f_i^* x\}, \{\phi_{ij}\}), (\{f_i^* y\}, \{\psi_{ij}\})).$$

So suppose we are an arrow of objects with descent data $\{\alpha_i\}\colon (\{f_i^* x\}, \{\phi_{ij}\}) \to (\{f_i^* y\}, \{\psi_{ij}\})$, that is a collection of arrows $\alpha_i\colon f_i^* x \to f_i^* y$ such that $\psi_{ij} \circ \mathrm{pr}_2^* \alpha_j = \mathrm{pr}_1^* \alpha_i \circ \phi_{ijp}$. Then there is a unique morphism $\alpha\colon x \to y$ such that $f_i^* \alpha = \alpha_i$. If we use the isomorphisms to change the domain to $(f_j \mathrm{pr}_2)^* x$ and the codomain to $(f_j \mathrm{pr}_2)^* y$, then we get $\mathrm{pr}_2^* \alpha_j = \mathrm{pr}_1^* \alpha_i$. So if we have locally defined morphisms, which are compatible overlaps, then they glue to a unique global morphism. This is precisely point 2 of our wishlist.

Definition 3.2 ([6, Def. 4.6])**.** A prestack $\pi\colon \mathcal{D} \to \mathcal{C}$ is a *stack* if for all object $C \in \mathcal{C}$ and all covering families $\mathcal{U}$ of C, the functor $\pi_C \to \pi_{\mathcal{U}}$ is essentially surjective.

Remark 3.3. This says precisely that for a stack the functor $\pi_C \to \pi_{\mathcal{U}}$ is an equivalence of categories.

Let us again spell out what this means in more concrete terms: Given an object with descent data $(\{\xi_i\}, \{\phi_{ij}\}) \in \pi_{\mathcal{U}}$, there is an object $x \in \pi_C$ such that $(\{f_i^* x\}, \{\psi_{ij}\})$ is isomorphic to the given object with descent data. So in particular $f_i^* x \cong \xi_i$ for all i. So if we have objects which are compatible on overlaps, then they glue to an essentially unique object x. But this is the weakened version of point 1 on our wishlist. So we have achieved our goal.

4 Examples

In this section we will discuss several examples of stacks. In all these examples, the stacks will be stacks living over the large site of all manifolds, Mfld. However, all examples can be developed in greater generality.

4.1 Representable stacks & sheaves

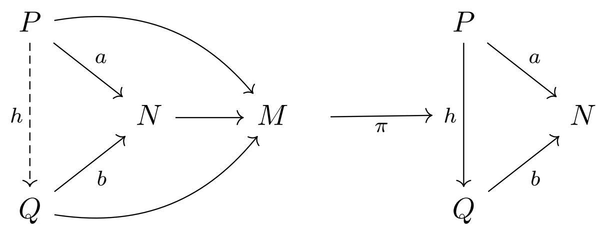
The representable stacks are perhaps amongst the simplest stacks on Mfld. However, they will play an important role when we start discussing *differentiable* stacks, since the subcategory of representable stacks is isomorphic to the category Mfld itself.

Let $M \in \mathsf{Mfld}$ and define $\pi\colon \mathsf{Mfld}_{/M} \to \mathsf{Mfld}$ by sending objects $g\colon N \to M$ to N, and morphisms $h\colon (P, f) \to (N, g)$ to $h\colon P \to N$. Then we have the following

Proposition 4.1. *The functor* $\pi\colon \mathsf{Mfld}_{/M} \to \mathsf{Mfld}$ *is a stack.*

Proof. We first have to show that $\pi\colon \mathsf{Mfld}_{/M} \to \mathsf{Mfld}$ is a category fibred in groupoids. To show that pullbacks exist, let $h\colon P \to N$ be a morphism in Mfld and let $g\colon N \to M \in \mathsf{Mfld}_{/M}$. Then $h\colon (P, gh) \to (N, g)$ is a pullback arrow projecting to h. Note that the pullback object is unique, namely $gh\colon P \to M$.

Next, we need to show that we can complete the following diagram



but it is clear, that picking h and augmenting the domain and codomain with the maps to M works.

Now, to show that this is fact a stack, we note that for any manifold $N \in \mathsf{Mfld}$ we have $\pi_N = \mathrm{Hom}_{\mathsf{Mfld}}(N, M)$, where this set is viewed as a discrete groupoid. Let $\mathcal{U} = \{f_i \colon U_i \to N\}$ be a covering family. Then we note that the functor $\pi_N \to \pi_{\mathcal{U}}$ is fully faithful since

$$\mathrm{Hom}_{\pi_N}(f, g) = \begin{cases} \{1_N\} & f = g \\ \emptyset & \text{otherwise} \end{cases}$$

and

$$\mathrm{Hom}_{\pi_{\mathcal{U}}}((\{f \circ f_i\}, \{1_{U_{ij}}\}), (\{g \circ f_i\}, \{1_{U_{ij}}\})) = \begin{cases} \{\{1_{U_i}\}\} & f = g \\ \emptyset & \text{otherwise} \end{cases}$$

To see that the functor is also essentially surjective, let $(\{g_i\}, \{1_{U_{ij}}\})$ be an object with descent data in $\pi_{\mathcal{U}}$. Let us define $g \colon N \to M$ by $g(x) = g_i(f_i^{-1}(x))$ if $x \in f_i(U_i)$. This well-defined: Suppose that $x \in f_j(U_j)$ as well, then there is a unique $(y, z) \in U_{ij} = U_i \times_N U_j$ (using the injectivity of f_i and f_j) such that $f_i(y) = f_j(z) = x$ and then it immediately follows that $g_i(y) = g_j(z)$, since the compatibility isomorphism is the identity. So $g \colon N \to M$ is a well-defined function of sets. Now, $\{f_i(U_i)\}$ is an open cover of M and $g|_{f_i(U_i)} = g_i \circ f_i^{-1}$ is smooth for each i. So it follows that g is in fact smooth. Finally, this last identity also shows that $g \circ f_i = g_i$. □

The proof above only used that $\mathrm{Hom}_{\mathsf{Mfld}}(-, M)$ is a sheaf, and one might suspect that all sheaves on a site give rise to a stack on the same site. This is indeed the case.

Theorem 4.2 ([6, Prop. 4.8])**.** *Let $\mathcal{C}$ be a site, and $F \colon \mathcal{C}^{\mathrm{op}} \to \mathsf{Set}$ a presheaf. This can also be viewed as a category fibred in (discrete) groupoids $\pi \colon (* \downarrow F) \to \mathcal{C}$ (here $* \downarrow F$ is the category of elements of F). Then F is a sheaf if and only if $\pi \colon (* \downarrow F) \to \mathcal{C}$ is a stack.*

4.2 Vector bundles & principal G-bundles

Let us now use this theory to study vector bundles. To this end, let us define the category $\mathsf{VB_n}$ with objects rank n vector bundles $p \colon E \to M$ and morphisms vector bundle morphisms covering a smooth map of the bases which are fibrewise linear isomorphisms.[2] We have a forgetful functor $\pi \colon \mathsf{VB}_n \to \mathsf{Mfld}$ by sending the vector bundles to their base, and the morphisms to the base map.

Lemma 4.3. *The functor $\pi \colon \mathsf{VB}_n \to \mathsf{Mfld}$ is a category fibred in groupoids.*

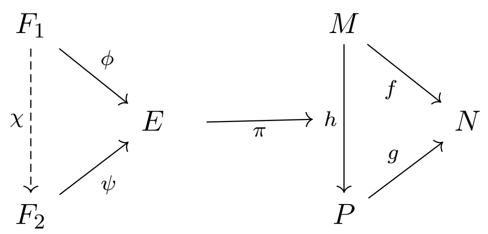
[2]This restriction of all vector bundle morphisms is necessary to ensure that π becomes fibred *in groupoids.*

Proof. Given a map $f\colon N \to M$ in Mfld and an object $p\colon E \to M \in \mathsf{VB}_n$, there must exist an object an morphism VB_n projecting to (covering) f. Such a fibre bundle is precisely given by the pullback bundle, which is defined as follows:

$$q\colon f^*E = \{(x,v) \in N \times E\colon p(v) = f(x)\} \to N, (x,v) \mapsto x$$

and with vector bundle morphism $\tilde{f}\colon f^*E \to E, (x,v) \mapsto v$. Then we see immediately that this is a fibrewise linear isomorphism, and so $\tilde{f}$ is a morphism in VB_n.

Next we need to show that if we have the following situation



then there is a unique morphism $\chi\colon F_1 \to F_2$ in VB_n completing the left triangle. We construct this χ as follows. We have a map $\tilde{\phi}\colon F_1 \to f^*E$ given by $v \mapsto (p_1(v), \phi(v))$. This is vector bundle morphism covering the identity which is a fibrewise isomorphism, hence this is a vector bundle isormorphism. Similarly, we have a vector bundle isormorphism $\tilde{\psi}\colon F_2 \to g^*E$ given by $v \mapsto (p_2(v), \psi(v))$. Next, we have a map $\bar{\chi}\colon f^*E \to g^*E, (x,v) \mapsto (h(x), v)$. This map is well-defined since $f = gh$. Moreover, this map is a fibrewise isomorphism, so a morphism in VB_n. Now, we define $\chi = \tilde{\psi}^{-1} \circ \bar{\chi} \circ \tilde{\phi}$. Then we have for $v \in F_1|_x$

$$\psi\chi(v) = \psi \circ \tilde{\psi}^{-1} \circ \bar{\chi}(x, \phi(v)) = \psi \circ \tilde{\psi}^{-1}(h(x), \phi(v)) = \psi(\psi^{-1}(\phi(v))) = \phi(v),$$

where in the penultimate step, ψ^{-1} means the fibre inverse on the fibre over $gh(x)$. So χ is one map making this triangle commute. The uniqueness follows from the fact that all maps are fibrewise isomorphisms and cover given maps. □

Next, let us recall that we can glue compatible locally defined maps between vector bundles together, as is customary in differential geometry.

Lemma 4.4. *Let $p\colon E \to M$, $q\colon F \to M$ be two vector bundles. Let $\{U_i\}_{i\in I}$ be an open cover of M. Let $\phi_i\colon E|_{U_i} \to F|_{U_i}$ be vector bundle morphism for each $i \in I$. Suppose that $\phi_i|_{U_{ij}} = \phi_j|_{U_{ij}}\colon E|_{U_{ij}} \to F|_{U_{ij}}$, where $U_{ij} = U_i \cap U_j$. Then there exists a unique vector bundle morphism $\phi\colon E \to F$ such that $\phi|_{U_i} = \phi_i$.*

Proof. We define $\phi\colon E \to F$ by $\phi(v) = \phi_i(v)$ if $p(v) \in U_i$. This is well-defined, since if $p(v) \in U_j$ as well, then $p(v) \in U_{ij}$ and so $\phi_i(v) = \phi_j(v)$. So we have defined a function. Moreover, it follows immediately from the definition that ϕ is linear on the fibres of E, since the ϕ_i are.

The collection $\{p^{-1}(U_i)\}$ is an open cover of E and $\phi|_{U_i} = \phi|_{p^{-1}(U_i)} = \phi_i$, which is smooth, so ϕ is a smooth map and satisfies the last condition. So we conclude that ϕ is the desired vector bundle morphsim.

Uniqueness of ϕ follows immediately from the prescribed local form. □

Corollary 4.5. *Under the hypothesis of the previous lemma, if all ϕ_i are vector bundle isomorphisms, then ϕ is a vector bundle isomorphism.*

Proof. A vector bundle morphism is an isomorphism if it is a fibrewise linear isomorphism. This is a local condition and so the result follows immediately from the extra hypothesis. □

Next, we turn to gluing vector bundles themselves together. Next, we can simplify our problem, by making identifications.

Lemma 4.6. *Let $f\colon U \to M$ be an open embedding, and let $p\colon E \to M$ be a vector bundle. Then the pullback map $\tilde{f}\colon f^*E \to E$ restricts to a vector bundle isomorphism $f^*E \to p^{-1}(f(U))$.*

Proof. We know that $\tilde{f}\colon f^*E \to E$ is a fibrewise linear isomorphism, so if we can show that it is injective and its image is $p^{-1}(f(U))$, then we are done.

Suppose that $\tilde{f}(x,v) = \tilde{f}(y,w)$, then we know that $f(x) = f(y)$, so $x = y$ since f is an embedding, and so also $v = w$. So $\tilde{f}$ is an injective map.

Let $v \in p^{-1}(f(U))$, then $p(v) = f(x)$ for some $x \in U$. Then $\tilde{f}(x,v) = v$. And for an arbitrary $(x,v) \in f^*E$ we have $x \in U$ and so $p(v) = f(x) \in f(U)$. So we are done. □

Lemma 4.7. *Let $f\colon U \to M$ be an open embedding, and let $p\colon E \to U$ be a vector bundle. Then $fp\colon E \to f(U)$ is a vector bundle.*

Proof. This follows from the fact that $f(U)$ is open, and $f\colon U \to f(U)$ is a diffeomorphism. □

Lemma 4.8. *Let M be a manifold, let $\{f_i\colon U_i \to M\}$ be a covering family in the large site of manifolds. Let $\{p_i\colon E_i \to U_i\}$ be a collection of vector bundles and assume that for all i,j there is an isomorphism $\phi_{ij}\colon \mathrm{pr}_2^*E_j \to \mathrm{pr}_1^*E_i$ such that the cocycle condition $\mathrm{pr}_{13}^*\phi_{ik} = \mathrm{pr}_{12}^*\phi_{ij} \circ \mathrm{pr}_{23}^*\phi_{jk}$ is satisfied. Then there is a vector bundle $p\colon E \to M$ such that $f_i^*E \cong E_i$.*

Proof. By the lemmas above we may reduce to following situation: We are given an open cover $\{U_i\}$ of M, vector bundles $\{p_i\colon E_i \to U_i\}$ and isomorphisms $\{\phi_{ij}\colon p_j^{-1}(U_i \cap U_j) \to p_i^{-1}(U_i \cap U_j)\}$ satisfying the cocycle condition.

Define $\widehat{E} = \coprod_{i\in I} E_i$. We say that two elements $v, w \in \widehat{E}$ are equivalent, $v \sim w$ if and only if $v \in E_i, w \in E_j$ and $\phi_{ij}(w) = v$. Then the cocycle condition ensures that this is an equivalence relation:

- Reflexivity follows from $\phi_{ii} = \phi_{ii} \circ \phi_{ii}$, so $\phi_{ii} = \mathrm{id}$.
- Symmetry follows from $\mathrm{id} = \phi_{ii} = \phi_{ij} \circ \phi_{ji}$, so $\phi_{ij}^{-1} = \phi_{ji}$.
- Transitivity follows from the cocycle condition directly.

So let $E = \widehat{E}/\sim$ as a set and define $p\colon E \to M, v \mapsto p_i(v)$ if $v \in E_i$. This projection map is well-defined since the ϕ_{ij} are vector bundle isomorphisms. We will show that this map is locally trivial, with trivializations which are smoothly compatible and fibrewise linear. Let $x \in M$ and pick an i such that $x \in U_i$ and pick an open $W_x \subset U_i$ such that there is a local trivialization $\bar{\Phi}_{i,x}\colon p_i^{-1}(W_x) \to W_x \times \mathbb{R}^n$ of E_i. Then we define $\Phi_x\colon p^{-1}(W_x) \to W_x \times \mathbb{R}^n, [v] \mapsto \bar{\Phi}_x(v)$, by picking a representative $v \in E_i$. This is a bijection.

Now, suppose that $x, y \in M$ are two points such that $W_x \cap W_y \neq \emptyset$. Let us say $W_x \subset U_i$ and $W_y \subset U_j$. Then on $(W_x \cap W_y) \times \mathbb{R}^n$ we have

$$\Phi_y \circ \Phi_x^{-1}(z,v) = \Phi_y([\bar{\Phi}_{i,x}^{-1}(z,v)]) = \Phi_y([\phi_{ji}(\bar{\Phi}_{i,x}^{-1}(z,v))]) = \bar{\Phi}_{j,y} \circ \phi_{ji} \circ \bar{\Phi}_{i,x}^{-1}(z,v).$$

This is a composition of smooth vector bundle isomorphisms, so smooth diffeomorphisms which are fibrewise linear isomorphisms. Then it follows from lemma 10.6 in [5] that $p\colon E \to M$ is a smooth vector bundle with the local trivializations as constructed above.

It remains to show that $p^{-1}(U_i) \cong E_i$ for all i. There is a natural map $\psi\colon E_i \to p^{-1}(U_i)$ given by $v \mapsto [v]$. This map is clearly surjective, and covers the identity. Therefore it suffices to show that it is smooth and a fibrewise linear isomorphism, which we will now do in one go. Let $x \in U_i$, then there is an open $W_x \subset V_j$ with trivialization Φ_x. We can restrict this trivialization to $p^{-1}(W_x \cap U_i)$. Then we see that

$$\Phi_x \circ \psi|_{p^{-1}(W_x \cap U_i)}(v) = \Phi_x([\phi_{ij}(v)]) = \bar{\Phi}_{j,x} \circ \phi_{ij}(v),$$

which is smooth and a fibrewise linear isomorphism. So ψ is a vector bundle isomorphism. □

Theorem 4.9. *The functor* $\pi\colon \mathsf{VB}_n \to \mathsf{Mfld}$ *is a stack.*

Proof. Lemma 4.3 shows that π is a category fibred in groupoids. Given a covering family $\mathcal{U}$ of $N \in \mathsf{Mfld}$, lemma 4.4 shows that $\pi_N \to \pi_{\mathcal{U}}$ is fully faithful, while lemma 4.8 shows that it is essentially surjective. □

4.2.1 Classification of vector bundles

Let us now use this stack to study the classification of vector bundles over a given manifold. For this, we need the following standard result in vector bundle theory.

Theorem 4.10 ([4, Theorem 4.7])**.** *Let $f, g\colon N \to M$ be two homotopic maps, and let $p\colon E \to M$ be a vector bundle. Then g^*E and f^*E are isomorphic. In particular, the only vector bundle up to isomorphism over a contractible manifold is the trivial vector bundle.*

Studying isomorphism classes of vector bundles over M is the same determining the skeleton of the groupoid π_M. Since π is a stack, we know that π_M is equivalent to $\pi_{\mathcal{U}}$ for covering families $\mathcal{U}$ of M. So if we choose a "smart" cover, we can use $\pi_{\mathcal{U}}$ to understand the skeleton of π_M.

Definition 4.11 (Adapted from [2, p. 42])**.** A cover $\mathcal{U} = \{U_i\}_{i\in I}$ of a manifold M is called *good* if all the U_i are contractible, as well as all finite intersections of elements of $\mathcal{U}$.

Proposition 4.12 ([2, Theorem 5.1])**.** *Every manifold admits a good cover.*

Let $\mathcal{U} = \{U_i\}_{i\in I}$ be a good cover. Then we know that each π_{U_i} is a connected groupoid, since each vector bundle over U_i is trivial. Let $(\{\xi_i\}, \{\phi_{ij}\})$ be an object with descent data. For each i, let $\alpha_i\colon \xi_i \to U_i \times \mathbb{R}^n$ be a vector bundle isomorphism. Define $\psi_{ij} = \mathrm{pr}_1^*\alpha_i \circ \phi_{ij} \circ (\mathrm{pr}_2^*\alpha_j)^{-1}$. Then it is an easy check that $\{\alpha_i\}\colon (\{\xi_i\}, \{\phi_{ij}\}) \to (\{U_i \times \mathbb{R}^n\}, \{\psi_{ij}\})$ is an isomorphism of objects with descent data.

Now, consider an object with descent data $(\{U_i \times \mathbb{R}^n\}, \{\psi_{ij}\})$. There is a one-to-one correspondence between vector bundle isomorphisms $\psi_{ij}\colon U_{ij} \times \mathbb{R}^n \to U_{ij} \times \mathbb{R}^n$ and smooth maps $\tau_{ij}\colon U_{ij} \to \mathrm{GL}_n(\mathbb{R})$. Under this identification, the cocycle condition of an object with descent data corresponds with $\tau_{ik} = \tau_{ij}\tau_{jk}$ on U_{ijk}. So an object with descent data $(\{U_i \times \mathbb{R}^n\}, \{\psi_{ij}\})$ is the same as a Čech cocycle $\{\tau_{ij}\}$ corresponding to the cover $\mathcal{U}$.

Next, let us consider an isomorphism of object with descent data $\{\alpha_i\}\colon (\{U_i \times \mathbb{R}^n\}, \{\phi_{ij}\}) \to (\{U_i \times \mathbb{R}^n\}, \{\psi_{ij}\})$. Then with the same identification, we can view $\alpha_i\colon U_i \to \mathrm{GL}_n(\mathbb{R})$.

Consider the diagram

$$\begin{array}{ccc} U_{ij} \times \mathbb{R}^n & \xrightarrow{\alpha_j} & U_{ij} \times \mathbb{R}^n \\ {\scriptstyle \phi_{ij}}\downarrow & & \downarrow{\scriptstyle \psi_{ij}} \\ U_{ij} \times \mathbb{R}^n & \xrightarrow[\alpha_i]{} & U_{ij} \times \mathbb{R}^n \end{array}$$

Chasing an element $(x, v) \in U_{ij} \times \mathbb{R}^n$ around this diagram, we obtain

$$(x, \alpha_i(x)\phi_{ij}(x)v) = (x, \psi_{ij}(x)\alpha_j(x)v),$$

where we identify the identification isomorphism with their corresponding maps $U_{ij} \to \mathrm{GL}_n(\mathbb{R})$. So we have that $(\{U_i \times \mathbb{R}^n\}, \{\phi_{ij}\}), (\{U_i \times \mathbb{R}^n\}, \{\psi_{ij}\})$ are isomorphic if and only if there is a collection $\{\alpha_i : U_i \to \mathrm{GL}_n(\mathbb{R})\}$ of smooth maps such that

$$\phi_{ij} = \alpha_i^{-1}\psi_{ij}\alpha_j,$$

that is, if and only if, $\{\phi_{ij}\}$ and $\{\psi_{ij}\}$ are equivalent as Čech cocycles, i.e. $\{\alpha_i\}$ is a Čech coboundary. So we have (re)discovered that vector bundles over M are classified by Čech cocycles up to equivalence on a good cover. That is, we have proven the following.

Theorem 4.13. *Let M be a manifold. There is a one-to-one correspondence*

$$\{\textit{Rank } n \textit{ vector bundles } E \to M\} \cong \frac{\textit{Čech cocycles } \{\phi_{ij}\}}{\textit{Čech coboundaries } \{\alpha_i\}}$$

4.2.2 Principal G-bundles and quotient stacks

In a previous lecture, we already saw that principal G-bundles over manifolds form a category fibred in groupoids $\pi : \mathsf{B}G \to \mathsf{Mfld}$. One can adapt the proofs in the previous section to show that this is also a stack. We then obtain the classifying stack of principal G-bundles over manifolds. In fact, assiging to a vector bundle its frame bundle, and conversely to a principal $\mathrm{GL}_n(\mathbb{R})$-bundle the associated vector bundle (where we take the standard action of $\mathrm{GL}_n(\mathbb{R})$) we obtain an "isomorphism of stacks".[3]

More generally, let us consider an adaption of example 1.5 in [3]. Let G be a Lie group acting on a manifold M via $A : G \times M \to M$. Then we define the quotient stack $[M/G]$ as the fibred category over Mfld with objects pairs $(p : P \to Y, f : P \to M)$ where $p : P \to Y$ is a principal G-bundle and $f : P \to M$ is a G-equivariant map. Morphisms between pairs $(p : P \to Y, f : P \to M)$ and $(q : Q \to Z, g : Q \to M)$ are given by smooth maps G-equivariant maps $\phi : P \to Q$ such that $g\phi = f$. The projection map $\pi : [M/G] \to \mathsf{Mfld}$ is given by sending $(p : P \to Y, f : P \to M)$ to Y and a morphism ϕ to the base map it covers. In the next section we will apply this construction to a familiar stack.

4.3 Triangles

In the first lecture we saw the classifying stack of triangles, based on chapter one of [1]. At the end of this lecture we saw that have a fine moduli object for triangles we needed to consider the set

$$N = \left\{(a, b, c) \in \mathbb{R}^3 : a + b + c = 2; a, b, c < 1\right\}.$$

[3]The precise meaning of this, is the content of a later lecture. Perhaps even neater is that you actually discover the frame bundle (if you did not know it before), when trying to show that the stack $\pi : \mathsf{VB}_n \to \mathsf{Mfld}$ is a differentiable stack.

endowed with an action of the dihedral group D_3 encoding the symmetries of triangles. A family of triangles over M then is a principal D_3-bundle $P \to M$ together with a D_3-equivariant map from P to N. So by the above, we see that the classifying stack of triangles is the quotient stack $[N/D_3]$.

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